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THE DISTRIBUTION OF STRESS NEAR THE TIP OF A CRACK WHICH ORIGINATES AT THE EDGE OF A CIRCULAR HOLE

by

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by

John Tweed
(University of Glasgow)

March 28, 1972

Preface

This report by Dr. John Tweed is the outgrowth of some of the author's work on fracture mechanics initiated while he was a Visiting Assistant Professor at North Carolina State University in 1969-70.

Although the major portion of this work was not supported by Grant AFOSR-69-1779, it is being presented as a project report as a part of our continuing interest in problems of fracture mechanics.

Dr. Tweed plans to obtain some computational results using the theoretical tools derived in this report.

W. J. Harrington

W. J. Harrington
Project Director

ABSTRACT

In this paper, the author makes use of a recent development in the theory of Mellin transforms to show that the stress intensity factor and the crack energy of a crack, which originates at the edge of a circular hole in an infinite elastic solid, are related in a simple fashion to the solution of a Fredholm integral equation of the second kind.

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1. INTRODUCTION.

The problem of determining the distribution of stress near the tip of a crack which originates at the edge of a circular hole in an infinite elastic solid appears to have been considered first by O. L. Bowie [1] who solves it by using a complex mapping technique. It would seem, however, that the results given by Bowie are not very accurate, so in this paper we wish to show that the stress intensity factor and crack energy are related to the solution of a Fredholm equation and may therefore be calculated to a high degree of accuracy.

We shall assume that the problem is to be solved under the conditions of plane strain and that the crack and the hole are defined, in plane polar coordinates (r, θ), by the relations $R \leq r \leq Rb$, $\theta = 0$ and $0 \leq r \leq R$, $0 \leq \theta \leq 2\pi$ respectively.

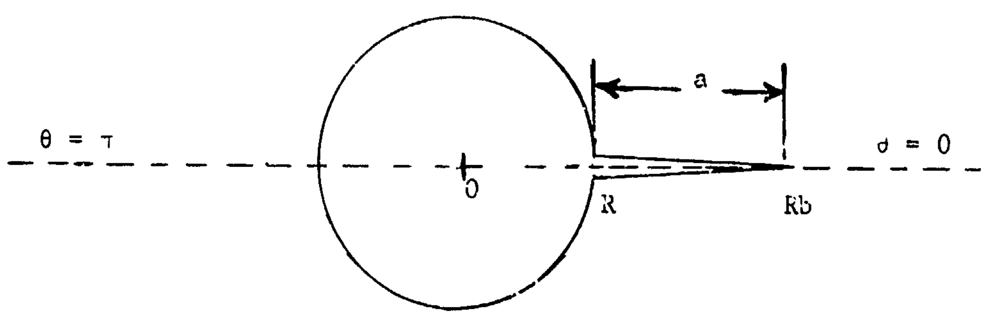

Figure 1.

If the loading is symmetric about the plane of the crack it is clear that the problem may be reduced to that of finding a solution of the equations of elasticity for the region $R < r < \infty$, $0 < \theta < \pi$, which is such that

(1) at infinity the stresses $\sigma_{rr}(r, \theta)$, $\sigma_{r\theta}(r, \theta)$, $\sigma_{\theta\theta}(r, \theta)$ are $O(r^{-2})$ and the displacements $u_r(r, \theta)$, $u_\theta(r, \theta)$ are $O(r^{-1})$,

(2) $\sigma_{r\theta}(r, 0) = 0$, $R < r < \infty$,

(3) $\sigma_{r\theta}(r, \pi) = u_\theta(r, \pi) = 0$, $R < r < \infty$,

(4) $\sigma_{rr}(R, \theta) = 0$, $0 < \theta < \pi$,

(5) $\sigma_{r\theta}(R, \theta) = 0$, $0 < \theta < \pi$,

(6) $\sigma_{\theta\theta}(r, 0) = - f(r)$, $R < r < Rb$,

(7) $u_\theta(r, 0) = 0$, $Rb < r < \infty$,

and

(8) $\lim_{r \to R_+} \dfrac{\partial u_\theta(r,0)}{\partial r} < \infty$.

2. REDUCTION OF THE PROBLEM TO AN INTEGRAL EQUATION.

In order to find a suitable representation for the stresses and displacements in the problem set out above we shall begin by superimposing the solutions of problems 1 and 2 below.

PROBLEM 1. Find a solution of the equations of elasticity for the region $R < r < \infty$, $0 < \theta < \pi$, which is such that

(a) at infinity the stresses are $O(r^{-2})$ and the displacements are $O(r^{-1})$,

(b) $\sigma_{r\theta}(r, 0) = u_\theta(r, 0) = 0$, $R < r < \infty$,

and

(c) $\sigma_{r\theta}(r, \pi) = u_\theta(r, \pi) = 0$, $R < r < \infty$.

By the method of separation of variables it is not difficult to show (e.g. see Durelli, Phillips and Tsao [?]) that the Airy stress function for this problem is given by

$$\psi(r, \theta) = c_0 \log r + c_1 r^{-1} \cos \theta + \sum_{n=2}^{\infty} [c_n r^{-n} + d_n r^{-n+2}] \cos n\theta , \qquad (2.1)$$

and the corresponding stresses and displacements by

$$\sigma_{rr}(r, \theta) = c_o r^{-2} - \sum_{n=1}^{\infty} [n(n + 1)c_n r^{-n-2} + (n + 2)(n - 1)d_n r^{-n}]\cos n\theta , \tag{2.2}$$

$$\sigma_{r\theta}(r, \theta) = - \sum_{n=1}^{\infty} [n(n + 1)c_n r^{-n-2} + n(n - 1)d_n r^{-n}] \sin n\theta , \tag{2.3}$$

$$\sigma_{\theta\theta}(r, \theta) = - c_o r^{-2} + \sum_{n=1}^{\infty} [n(n + 1)c_n r^{-n-2} + (n - 2)(n - 1)d_n r^{-n}]\cos n\theta , \tag{2.4}$$

$$u_r(r, \theta) = \frac{1 + \eta}{E}\Big\{- c_o r^{-1} + c_1 r^{-2}\cos \theta + \sum_{n=2}^{\infty} [n\, c_n r^{-n-1} + (n + 2 - 4\eta)d_n r^{-n+1}]\cos n\theta\Big\} , \tag{2.5}$$

and

$$u_\theta(r, \theta) = \frac{1 + \eta}{E}\Big\{c_1 r^{-2}\sin\theta + \sum_{n=2}^{\infty} [n\, c_n r^{-n-1} + (n - 4 + 4\eta)d_n r^{-n+1}]\sin n\theta\Big\}, \tag{2.6}$$

where E is the Young's modulus and η is the Poisson's ratio of the material.

PROBLEM 2. Find a solution of the equations of elasticity, for the half-plane $0 \le r \le \infty$, $0 < \theta < \pi$, which is such that

(a) at infinity the stresses are $O(r^{-2})$ and the displacements are $O(r^{-1})$,

(b) at the origin the stresses and displacements are bounded,

(c) $\sigma_{r\theta}(r, 0) = 0$, $0 \le r < \infty$,

and

(d) $\sigma_{r\theta}(r, \pi) = u_\theta(r, \pi) = 0$, $0 \le r < \infty$.

By utilising the properties of the Mellin transform (e.g. see Tranter [3]) it can be shown that the solution of the problem may be written in the form

$$\sigma_{rr}(r, \theta) = r^{-2}M^{-1}\left[\frac{A(s)}{2 \sin\pi s}\Big\{(s + 4)\cos(\theta - \pi)(s + 2) - (s + 2)\cos(\theta - \pi)s\Big\} ;\ r\right] , \tag{2.7}$$

$$\sigma_{r\theta}(r, \theta) = r^{-2}M^{-1}\left[\frac{(s + 2)A(s)}{2 \sin \pi s} \{\sin(\theta - \pi)(s + 2) - \sin(\theta - \pi)s\} ; r\right], \quad (2.8)$$

$$\sigma_{\theta\theta}(r, \theta) = r^{-2}M^{-1}\left[\frac{A(s)}{2 \sin\pi s} \{(s + 2)\cos(\theta - \pi)s - s \cos(\theta - \pi)(s + 2)\} ; r\right], \quad (2.9)$$

$$u_r(r, \theta) = \frac{1 + \eta}{rE} M^{-1}\left[\frac{A(s)}{2(s + 1)\sin\pi s} \{(s + 2)\cos(\theta - \pi)s - (s + 4 - 4\eta)\cos(\theta - \pi)(s + 2)\} ; r\right], \quad (2.10)$$

and

$$u_\theta(r, \theta) = \frac{1 + \eta}{rE} M^{-1}\left[\frac{A(s)}{2(s + 1)\sin\pi s} \{(s + 2)\sin(\theta - \pi)s - (s - 2 + 4\eta)\sin(\theta - \pi)(s + 2)\} ; r\right], \quad (2.11)$$

where M^{-1} is the inverse Mellin transform and $-1 < \mathrm{Re}(s) < 0$.

Clearly by superimposing the solutions of these two problems we obtain a solution of the equations of elasticity for the region $R < r < \infty$, $0 < \theta < \pi$ which automatically satisfies conditions (1), (2), and (3) and which is such that

$$\sigma_{rr}(r, \theta) = c_o r^{-2} - \sum_{n=1}^{\infty} [n(n + 1)c_n r^{-n-2} + (n + 2)(n - 1)d_n r^{-n}]\cos n\theta + r^{-2}M^{-1}\left[\frac{A(s)}{2 \sin\pi s} \{(s + 4)\cos(\theta - \pi)(s + 2) - (s + 2)\cos(\theta - \pi)s\} ; r\right], \quad (2.12)$$

$$\sigma_{r\theta}(r, \theta) = - \sum_{n=1}^{\infty} \left[n(n + 1)c_n r^{-n-2} + n(n - 1)d_n r^{-n}\right]\sin n\theta + r^{-2}M^{-1}\left[\frac{(s + 2)A(s)}{2 \sin\pi s} \{\sin(\theta - \pi)(s + 2) - \sin(\theta - \pi)s\} ; r\right], \quad (2.13)$$

$$\sigma_{\theta\theta}(r,\theta) = -c_o r^{-2} + \sum_{n=1}^{\infty}[n(n+1)c_n r^{-n-2} + (n-2)(n-1)d_n r^{-n}]\cos n\theta$$

$$+ r^{-2}M^{-1}\left[\frac{A(s)}{2\sin\pi s}\{(s+2)\cos(\theta-\pi)s\right.$$

$$\left. - s\cos(\theta-\pi)(s+2)\} \; ; \; r\right], \tag{2.14}$$

$$u_r(r,\theta) = \frac{1+\eta}{E}\left[-c_o r^{-1} + c_1 r^{-2}\cos\theta + \sum_{n=2}^{\infty}\{nc_n r^{-n-1} + (n+2-4\eta)d_n r^{-n-1}\}\cos n\theta\right]$$

$$+ \frac{1+\eta}{rE}M^{-1}\left[\frac{A(s)}{2(s+1)\sin\pi s}\{(s+2)\cos(\theta-\pi)s\right.$$

$$\left. - (s+4-4\eta)\cos(\theta-\pi)(s+2)\} \; ; \; r\right], \tag{2.15}$$

and

$$u_\theta(r,\theta) = \frac{1+\eta}{E}\left[c_1 r^{-2}\sin\theta + \sum_{n=2}^{\infty}\{nc_n r^{-n-1} + (n-4+4\eta)d_n r^{-n+1}\}\sin n\theta\right]$$

$$+ \frac{1+\eta}{rE}M^{-1}\left[\frac{A(s)}{2(s+1)\sin\pi s}\{(s+2)\sin(\theta-\pi)s\right.$$

$$\left. - (s-2+4\eta)\sin(\theta-\pi)(s+2)\} \; ; \; r\right], \tag{2.16}$$

where $-1 < \mathrm{Re}(s) < 0$. The complete solution of the problem may now be obtained by choosing the unknown function A(s) and the unknown sequences $\{c_n\}$ and $\{d_n\}$ in such a way that the remaining boundary conditions are satisfied.

From (2.14) and (2.16) we see that conditions (6) and (7) will be satisfied if A(s) is a solution of the dual equations

$$\begin{aligned} &M^{-1}[A(s)\cot\pi s \; ; \; r] = -r^2 f(r) - r^2 F(r), \quad R < r < Rb \\ &M^{-1}[(1+s)^{-1}A(s);r] = 0 \qquad\qquad , \quad Rb < r < \infty \end{aligned} \tag{2.17}$$

where $-1 < \mathrm{Re}(s) < 0$ and

$$F(r) = -c_o r^{-2} + \sum_{n=1}^{\infty} [n(n+1)c_n r^{-n-2} + (n-2)(n-1)d_n r^{-n}]. \tag{2.18}$$

If we now assume that $A(s)$ may be written in the form

$$A(s) = \int_R^{Rb} p(t)\, t^{s+1} dt \tag{2.19}$$

we find that (see Tweed [4])

$$M^{-1}[A(s)(1+s)^{-1}; r] = r \begin{cases} 0, & 0 < r < R \\ \int_r^{Rb} p(t)dt, & R < r < Rb \\ 0, & Rb < r < \infty \end{cases} \tag{2.20}$$

and

$$M^{-1}[A(s) \cot \pi s \; ; r] = \frac{r}{\pi} \int_R^{Rb} \frac{t\, p(t)}{t-r}\, dt \tag{2.21}$$

and hence that the equations (2.17) will be satisfied if

$$\frac{1}{\pi} \int_R^{Rb} \frac{t\, p(t)\, dt}{t-r} = -\, r\, f(r) - r\, F(r), \quad R < r < Rb. \tag{2.22}$$

The equation (2.22) is well known and Tricomi [5] has shown that its solution is given by

$$t\, p(t) = \frac{1}{\pi} \left(\frac{t-R}{Rb-t} \right)^{1/2} \int_R^{Rb} \left(\frac{Rb-y}{y-R} \right)^{1/2} \frac{y\, f(y) + yF(y)}{y-t}\, dy \tag{2.23}$$

$$+ \frac{C}{[(Rb-t)(t-R)]^{1/2}},$$

where C is an arbitrary constant. In order to determine C we make use of condition (8) which together with (2.16) and (2.20) clearly implies that $\lim_{r \to R_+} p(r)$ exists and hence that $C = 0$. It now follows that

$p(t)$ is given by the expression

$$p(t) = \frac{1}{\pi t}\left(\frac{t - R}{Rb - t}\right)^{1/2} \int_R^{Rb} \left(\frac{Rb - y}{y - R}\right)^{1/2} \frac{y\, f(y) + y\, F(y)}{y - t}\, dy, \tag{2.23}$$

Similarly, on applying conditions (4) and (5) we find that

$$- c_o + \sum_{n=1}^{\infty} [n(n + 1)c_n R^{-n} + (n + 2)(n - 1)d_n R^{-n+2}]\cos n\theta$$

$$= M^{-1}\left[\frac{A(s)}{2 \sin\pi s} \{(s + 4)\cos(\theta - \pi)(s + 2) - (s + 2)\cos(\theta - \pi)s\} \ ; \ R\right],$$

$$\sum_{n=1}^{\infty} [n(n + 1)c_n R^{-n} + n(n - 1)d_n R^{-n+2}]\sin n\theta$$

$$= M^{-1}\left[\frac{A(s)(s + 2)}{2 \sin\pi s} \{\sin(\theta - \pi)(s + 2) - \sin(\theta - \pi)s\} \ ; \ R\right]$$

where $0 < \theta < \pi$, and hence that

$$c_o = \frac{1}{2\pi} M^{-1}\left[A(s) \left\{\frac{s + 2}{s} - \frac{s + 4}{s + 2}\right\} ; \ R\right], \tag{2.24}$$

$$c_n = - \frac{2R^n}{\pi} M^{-1}\left[\frac{A(s)(s + 2)}{(s^2 - n^2)(s + 2 + n)} \ ; \ R\right], \quad n \geq 1 \tag{2.25}$$

and

$$d_n = \frac{2R^{n-2}}{\pi} M^{-1}\left[\frac{A(s)\ (s + 2)}{(s + n)([s + 2]^2 - n^2)} \ ; \ R\right], \quad n \geq 2 \tag{2.26}$$

where $-1 < Re(s) < 0$.

On substituting from (2.19) into (2.24) through (2.26) and working out the inverse Mellin transforms, we now find that

$$c_o = - \frac{R^2}{\pi} \int_R^{Rb} t^{-1}p(t)dt , \tag{2.27}$$

$$c_n = \frac{R^n}{2\pi} \int_R^{Rb} t\ p(t) \left\{\frac{n}{n + 2} \left(\frac{R}{t}\right)^{n+2} - \frac{n - 2}{n} \left(\frac{R}{t}\right)^n\right\} dt, \quad n \geq 1 \tag{2.28}$$

and

$$d_n = \frac{R^{n-2}}{2\pi} \int_R^{Rb} t\, p(t) \left\{ \frac{n-2}{n-1} \left(\frac{R}{t}\right)^n - \left(\frac{R}{t}\right)^{n+2} \right\} dt, \quad n \geq 2 \tag{2.29}$$

and hence that

$$F(r) = \frac{1}{\pi} \int_R^{Rb} p(t)\, K(r, t)\, dt , \tag{2.30}$$

where

$$K(r, t) = \frac{R^2(R^2 - t^2)^2}{t(R^2 - rt)^3} - \frac{t(R^2 - t^2)}{(R^2 - rt)^2} - \frac{t}{R^2 - rt} + \frac{R^2 - t^2}{r^2 t} - \frac{1}{r} . \tag{2.31}$$

At this point we find it convenient to introduce the function $P(t)$ which is defined by the equation

$$P(t) = [(t - R)(Rb - t)]^{1/2} p(t). \tag{2.32}$$

On substituting from (2.30) into (2.23) and taking account of (2.32) we see that $P(t)$ must satisfy the integral equation

$$P(t) - \int_R^{Rb} \frac{P(\rho)\, M(t, \rho)}{[(\rho - R)(Rb - \rho)]^{1/2}}\, d\rho = S(t), \tag{2.33}$$

where

$$S(t) = \frac{t - R}{\pi t} \int_R^{Rb} \left(\frac{Rb - y}{y - R}\right)^{1/2} \frac{y\, f(y)\, dy}{y - t} , \tag{2.34}$$

and

$$M(t, \rho) = \frac{t - R}{\pi^2 t} \int_R^{Rb} \left(\frac{Rb - y}{y - R}\right)^{1/2} \frac{y\, K(y, \rho)}{y - t}\, dy. \tag{2.35}$$

If we now substitute from (2.31) into (2.35) we find that $M(t, \rho)$ may be written in the form

$$M(t, \rho) = \frac{t - R}{\pi t} \cdot \{\rho^{-4}R^2(R^2 - \rho^2)^2 J_3(t, R^2/\rho) + \rho^{-1}(R^2 - \rho^2)[J_2(t, 0) - J_2(t, R^2/\rho)]$$

$$+ J_1(t, 0) - J_1(t, R^2/\rho)\} , \qquad (2.36)$$

where

$$J_n(t, x) = \frac{1}{\pi} \int_R^{Rb} \left(\frac{Rb - y}{y - R}\right)^{1/2} \frac{y\,dy}{(y - t)(x - y)^n} , \qquad (2.37)$$

$0 \leq x < R < t < Rb, \quad n = 1, 2, 3.$

On making use of the result

$$J(x) = \frac{1}{\pi} \int_R^{Rb} \left(\frac{Rb - y}{y - R}\right)^{1/2} \frac{dy}{y - x}$$

$$= \begin{cases} \left(\dfrac{x - Rb}{x - R}\right)^{1/2} - 1 , & Rb < x < \infty \\ -1 , & R < x < Rb \\ \left(\dfrac{Rb - x}{R - x}\right)^{1/2} - 1 , & 0 \leq x < R \end{cases}$$

and the fact that

$$J_1(t, x) = (x - t)^{-1}[t\,J(t) - x\,J(x)] ,$$

$$J_2(t, x) = -\frac{\partial}{\partial x} J_1(t,x) \text{ and } J(t, x) = -\frac{1}{2}\frac{\partial}{\partial x} J_2(t,x)$$

It now becomes a simple matter to show that $M(t, \rho)$ is given by the formula

$$M(t, \rho) = \frac{(t - R)R^2(R^2 - \rho^2)^2}{\pi t} \left\{ \frac{(b - 1)\,t}{2R(R^2 - \rho t)^2(b\rho - R)^{1/2}(\rho - R)^{3/2}} \right.$$

$$\left. - \frac{(b - 1)[(1 + 3b)\rho - 4R]}{8\rho\,(R^2 - \rho t)(b\rho - R)^{3/2}(\rho - R)^{5/2}} - \frac{t\,(b\rho - R)^{1/2}}{\rho(R^2 - \rho t)^3(\rho - R)^{1/2}} \right\}$$

$$+ \frac{(t - R)(R^2 - \rho^2)}{\pi t} \left\{ \frac{\rho t\ (b\rho - R)^{1/2}}{(R^2 - \rho t)^2 (\rho - R)^{1/2}} \right.$$

$$\left. - \frac{R(b - 1)\rho}{2(R^2 - \rho t)(b\rho - R)^{1/2}(\rho - R)^{3/2}} - \frac{b^{1/2}}{\rho t} \right\}$$

$$+ \frac{(t - R)R^2(b\rho - R)^{1/2}}{\pi t(R^2 - \rho t)(\rho - R)^{1/2}} . \tag{2.38}$$

3. THE STRESS INTENSITY FACTOR AND THE CRACK ENERGY.

We shall now show that the stress intensity factor K and the crack energy W which are defined by the equations

$$K = - \lim_{r \to Rb_-} [2(Rb - r)]^{1/2} \frac{E}{2(1 - \eta^2)} \frac{\partial u_\theta}{\partial r}(r,0) \tag{3.1}$$

and

$$W = - \int_R^{Rb} \sigma_\theta(r,\ 0)\ u_\theta(r,\ 0)\ dr \tag{3.2}$$

respectively, are simply related to the function P(t) which was introduced in the last section.

By substituting from (2.32) into (2.20) and taking account of (2.16) we see that

$$u_\theta(r,\ 0) = - \frac{2(1 - \eta^2)}{E} \int_r^{Rb} \frac{P(t)\ dt}{[(t - R)(Rb - t)]^{1/2}} ,\ R \leq r \leq Rb. \tag{3.3}$$

It now follows that K is given by

$$K = - \frac{2^{1/2}}{[R(b - 1)]^{1/2}} P(Rb) \tag{3.4}$$

and W by

$$W = - \frac{2(1 - \eta^2)}{E} \int_R^{Rb} \frac{P(t)\ dt}{[(Rb - t)(t - R)]^{1/2}} \int_R^t f(r)\ dr. \tag{3.5}$$

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